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EVALUATION OF HYPERSINGULAR INTEGRALS IN THE BOUNDARY ELEMENT METHOD BY COMPLEX VARIABLE TECHNIQUES

CHUNG-YUEN HUI and SUBRATA MUKHERJEE
Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14850, U.S.A.

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Abstract—Strongly singular and hypersingular integrals that occur in many two dimensional applications of the boundary element method (e.g. potential theory and liner elasticity) can be treated in a unified manner by complex analysis. This paper presents an elegant approach for the evaluation of such hypersingular integrals. The limit to the boundary method, where a source point is moved from the inside (or outside) of a body to its boundary, is carried out using complex variables. The connection of this limit with the Hadamard Finite part of hypersingular integrals and principal value integrals is established. Many examples are worked out throughout the paper to illustrate the theory. Copyright © 1996 Published by Elsevier Science Ltd.

1. INTRODUCTION

Hypersingular integrals (HSI), their interpretations and roles in the boundary element method (BEM) have attracted considerable attention from researchers during recent years. A review article by Tanaka *et al.* (1994), for example, devoted to regularization techniques applied to the BEM, cites 350 references! Although this review article is concerned with regularization techniques for both strongly and HSI occurring in boundary integral equations (BIE), as well as those of nearly singular and nearly hypersingular integrals arising when a source point is near but not on a boundary element, the discussion of hypersingular integrals is still a major part of this article by Tanaka *et al.* A thorough and updated review of HSI can be found in a recently completed Ph.D. dissertation by Paulino (1995).

The various strategies employed by researchers for the numerical evaluation of hypersingular integrals, within the context of the BEM, are briefly summarised below. Without claiming generality, these methods can be classified into:

• Analytical integration followed by differentiation (e.g. Cruse 1974). This method is elegant but is usually possible for simple elements and shape functions, e.g. flat boundary elements for three dimensional BEM applications.

• Special quadrature rules (e.g. Kutt, 1975, Linz, 1985, Aliabadi and Hall, 1987, and Tsamasphyros, 1990). This method should be used with caution. It is often very difficult to derive or apply particular quadrature rules for higher dimensional (e.g. two dimensional) integrals and curved integration domains. In general, the numerical results for such cases are not sufficiently accurate.

• Integration by parts (e.g. Sladek and Sladek, 1984, Polch *et al.*, 1987, Nishimura and Kobayashi, 1989). This approach transfers derivatives from hypersingular kernels to functions multiplying them. The singularity of the kernel is weakened, at the expense of modifying the primary variables of a boundary integral equation (BIE). These same final formulae have been obtained by other approaches e.g. Zhang and Achenbach (1989).

• Use of special solutions (e.g. Rudolphi, 1990, 1991, Lutz *et al.*, 1992). This approach can be viewed as an extension of the well known use of rigid body motion solutions for the evaluation of strongly singular integrals in BEM. This is an elegant method but lacks general applicability (e.g., it fails when one has to deal with open surfaces.)

• Conversion (e.g. Krishnasamy *et al.*, 1990, Bonnet, 1989, Bonnet and Bui, 1993). The general idea (by Krishnasamy *et al.*, 1990) is subtraction and addition of relevant terms to the hypersingular BIE, followed by conversion to integrals, that are at most weakly singular, by employing Stokes' theorem. Bonnet and Bui (1993) carried out integration by parts, followed by first order regularization using a variant of Stokes' theorem.

• Series expansion of Integrand (Guigiani *et al.*, 1992). This approach is very general. Here a Laurent series expansion of the relevant integrand is followed by transformation into a parameter plane of intrinsic coordinates.

• Continuation approach (Rosen and Cormack, 1992, 1993). This approach is based on a limit procedure. Basically, singular integrals are viewed as "continuations" of non-singular ones. This method is quite general.

• Limit to the boundary (e.g. Gray 1989, Gray *et al.*, 1990, Gray and Soucie, 1993, Paulino, 1995). In this approach, a source point is first moved away from the boundary of a body. A careful limit process is then performed to bring the source point back to the boundary of the body. The resulting integrals are separated into regular and potentially hypersingular parts. The regular ones can be evaluated by usual Guassian quadrature while the potentially hypersingular ones are evaluated analytically, usually with the assistance of symbolic computation. This approach is very general.

An important topic, of general interest to the BEM community and others, is Hadamard finite parts (HFP) of integrals and their relationship to HSI. An excellent paper on the subject is Mangler (1952) in which the author presents a general definition of HFP integrals (his eqns 13, 14) and formulae for evaluating them. Mangler uses real variable calculus to derive his formulae. Toh and Mukherjee (1994) have recently proposed to interpret a HSI as a HFP integral for sufficiently smooth functions. This paper contains a theorem (p. 2306), relating the value of the "limit to the boundary" of a hypersingular integral, to its Hadamard finite part. This is basically a generalisation of previous work by Krishnasamy *et al.* (1990).

This paper is concerned with the limit to the boundary approach for the evaluation of HSI that occurs in the BEM. Specifically, we evaluate general integrals of the type shown in eqn (1). The focus of this paper is two dimensional applications of the hypersingular BEM. It turns out that all singular integrals that arise from the BEM in two dimensions can be treated in a unified away using complex variable theory. Indeed, integrals of the form

$$I_n^{\pm} = \lim_{z \to \zeta^{\pm}} \frac{1}{2\pi i} \int_C \frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}} \tag{1}$$

often arise in complex two dimensional hypersingular BIE formulations, for example, the Laplace equation or plane strain/stress elasticity (see, for example, Lee, 1993 for the elasticity problem). Here $f(t)$ is a sufficiently smooth complex valued function, $i = \sqrt{-1}$ and n is a positive integer. $z = x+iy$ and t are complex variables. C is a simple smooth open or closed curve in the x-y plane and ζ is a point on C. It is assumed that ζ is not an end point if C is open. C is considered smooth when it can be parameterised by

$$x = x(s), \quad y = y(s), \tag{2}$$

where $s \in [0, a]$. It is assumed that $x(s), y(s)$ have continuous first derivatives within $(0, a)$ and these derivatives are never simultaneously zero so that the tangent at any interior point of the curve exists. C is a closed curve if $x(0) = x(a)$ and $y(0) = y(a)$. In the rest of this work, we shall assume that C does not intersect itself and that s is the arc length measured from the point $z_o = (x(0), y(0))$. Also, C is to be traversed in the positive direction, which is defined to be counterclockwise. The hypersingular integral $I_n^{\pm}(\zeta)$ in (1) can be rewritten as

$$I_n^+(\zeta) \equiv \lim_{z \to \zeta^+} F_n(z) \tag{3a}$$

where $F_n(z)$ is the analytic function defined by

$$F_n(z) \equiv \frac{1}{2\pi i} \int_C \frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}} \tag{3b}$$

The notation $\lim_{z \to \zeta^+} F_n(x+iy)$ has the following meaning: if one orients oneself at the point ζ facing the direction of integration, then the limit of $F_n(z)$ as $z \to \zeta$ from the left and the right will be denoted by $I_C^+(\zeta)$ and $I_C^-(\zeta)$, respectively. If C is a closed contour, then $I_C^-(\zeta)$ and $I_C^-(\zeta)$ are the limits as $z \to \zeta$ from the inside and outside of the region Ω enclosed by C.

A simple way of understanding the origin of integrals such as eqn (1) is to observe that any solution of Laplace's equation $\nabla^2 w = 0$ in the region Ω enclosed by a closed contour C can be written as

$$w(x, y) = \mathrm{Re}\left[\frac{1}{2\pi i} \oint_C \frac{f(t)\,\mathrm{d}t}{t-z}\right] \quad (x, y) \in \Omega \tag{4}$$

where $f(t)$ is some unknown complex valued function. Integrals of the type in eqn (1) arise if one seeks to evaluate the gradient of w at a point on C by first differentiating (4) with respect to z and then taking the limit as z approaches the boundary point ζ.

The plan of this paper is as follows. To motivate the connection between HSI and BEM as well as the relation between our complex variable formulation and the usual two dimensional real variable formulation, the simple problem of Laplace equation in the upper half plane is considered in Section 2. In this case both t and $f(t)$ in eqn (1) are real and C is the real axis. Section 2 is followed (in Section 3) by a discussion of the evaluation of HSI of the type:

$$I_n^-(z) \equiv \lim_{y \to 0^+} \frac{1}{2\pi i} \int_a^b \frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}} \tag{5}$$

where a and b are real but $f(t)$ can be complex valued. In what is to follow, we shall be interested in the definition of the hypersingular integral

$$\hat{J}_1 \equiv \frac{1}{2\pi} \int_a^b \frac{f(t)}{(t-x)^2}\,\mathrm{d}t \quad x \in (a, b) \tag{6}$$

when $f(t)$ is real.

A word of caution! It might be tempting to define $\hat{J}_1 = iI_1$, i.e.,

$$\hat{J}_1 \equiv \lim_{y \to 0^\pm} \frac{1}{2\pi} \int_a^b \frac{f(t)}{(t-z)^2}\,\mathrm{d}t$$

This definition is inconsistent with (6) as can be seen from the discussion given below.

We first observe that,

$$\frac{1}{(t-z)^2} = \frac{(t-x)^2 - y^2}{[(t-x)^2 + y^2]^2} + i\frac{2(t-x)y}{[(t-x)^2 + y^2]^2}$$

It follows that, for real $f(t)$

$$\mathrm{Re}\, I_1^{\pm} = \lim_{y\to 0^{\pm}} \left[\frac{y}{2\pi} \int_a^b \frac{2(t-x)f(t)}{[(t-x)^2+y^2]^2} \mathrm{d}t \right] \tag{7}$$

$$\mathrm{Im}\, I_1^{\pm} = -\lim_{y\to 0^{\pm}} \left[\frac{1}{2\pi} \int_a^b \frac{[(t-x)^2-y^2]f(t)}{[(t-x)^2+y^2]^2} \mathrm{d}t \right] \tag{8}$$

where Re and Im denote, respectively, the real and imaginary part of the complex quantity $I_1^{\pm}$.

As will be proved later in the paper, $\mathrm{Re}\, I_1^{\pm}$ is not equal to zero, but equals $\pm\frac{1}{2}(\mathrm{d}f/\mathrm{d}x)$! The integral $\hat{J}_1$ (eqn (6)) is defined in this work to be

$$\hat{J}_1 = -\mathrm{Im}\, I_1^{\pm} \tag{9}$$

We shall also prove later in this paper that $\mathrm{Im} I_1^{\pm}$ does not depend on whether x is approached from above or below the x axis, whereas $\mathrm{Re} I_1^{\pm}$ does.

For the special case of $n = 0$, eqn (3) can be evaluated using the Plemelj or Sokhotski formulae (see Muskhelishvili, 1953). The central theme of this work is to generalise Plemelj formulae for $n \geqslant 1$. These new formulae are expected to be extremely useful in numerical implementation of two dimensional hypersingular BIEs. This paper closes with a discussion of the connection between the HFP integrals for the general denominator $(t-x)^{n+1}$ and the new formulae presented in this work. We mention here that, after submitting this manuscript for publication, we became awarer of a recent paper by Linkov and Mogilevskaya (1994). Their work also presents extended Plemelj formulae but is restricted to the case $n = 1$ in our eqn (3).

2. LAPLACE EQUATION IN THE UPPER HALF PLANE (UHP)

To motivate the connection between HSI and BEM, consider

$$\nabla^2 w = 0 \quad (x, y) \in \Omega, \tag{10}$$

where $\Omega = \{(x, y) | y > 0\}$ is the upper half plane. Assuming that w vanishes sufficiently fast at infinity (i.e., $w = O(r^{-1-\varepsilon})$, $\varepsilon > 0$, $r = \sqrt{x^2+y^2}$), the standard BIE is:

$$w(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[-\ln r(p, Q) \frac{\partial w(Q)}{\partial n(Q)} + w(Q) \frac{\partial \ln r(p, Q)}{\partial n(Q)} \right] \mathrm{d}\eta \tag{11}$$

where $p = (x, y)$ is a source point in Ω and $Q = (\eta, \xi = 0)$ is a field point on the real axis. Let

$$\tau(\eta) \equiv w_{,\xi}(\eta, \xi = 0) = -\frac{\partial w}{\partial n}(Q) \tag{12}$$

where $_{,\xi}$ denotes the partial derivative with respect to ξ. Equation (11) is:

$$w(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau(\eta) \ln \sqrt{(\eta-x)^2+y^2}\, \mathrm{d}\eta + \frac{y}{2\pi} \int_{-\infty}^{\infty} \frac{w(\eta)}{(\eta-x)^2+y^2} \mathrm{d}\eta \tag{13}$$

Equation (13) can be rewritten as:

$$w(x,y) = \mathrm{Re}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\tau(\eta)\ln(\eta-z)\,\mathrm{d}\eta\right] + \mathrm{Re}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(\eta)}{\eta-z}\mathrm{d}\eta\right] \tag{14}$$

where Re denotes the real part of a complex number.

Taking the limit as $y \to 0^+$ in eqn (14) gives

$$w(x) \equiv w(x, y=0^+) = \frac{1}{\pi}\int_{-\infty}^{\infty}\tau(\eta)\ln(\eta-x)\,\mathrm{d}\eta \tag{15}$$

where use is made of the Plemelj formula

$$\lim_{y\to 0^\pm}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\eta)}{\eta-z}\mathrm{d}\eta = \pm\frac{f(x)}{2} + \frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{f(\eta)}{\eta-x}\mathrm{d}\eta \tag{16}$$

and P denotes the Principal value integral. Note that the second term on the RHS of (13) is not zero but equals $w(x)/2$ since the function

$$\frac{1}{\pi}\frac{y}{[(\eta-x)^2+y^2]}$$

can be interpreted as the Dirac delta function $\delta(\eta-x)$ in the distribution sense. Also, note that the integral of the above function with respect to η, over $-\infty, \infty$, is equal to 1.

Suppose one is interested in the gradient of w as $z = x+iy \to x$ from Ω. For any point $(x, y)\in\Omega$, differentiation of (14) gives:

$$w_{,y}(x,y) = \mathrm{Re}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tau(\eta)}{\eta-z}\mathrm{d}\eta\right] - \mathrm{Im}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(\eta)}{(\eta-z)^2}\mathrm{d}\eta\right] \tag{17a}$$

$$w_{,x}(x,y) = \mathrm{Im}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tau(\eta)}{\eta-z}\mathrm{d}\eta\right] + \mathrm{Re}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(\eta)}{(\eta-z)^2}\mathrm{d}\eta\right] \tag{17b}$$

so that

$$w_{,x}(x,y) - iw_{,y}(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tau(\eta)}{\eta-z}\mathrm{d}\eta + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(\eta)}{(\eta-z)^2}\mathrm{d}\eta \tag{17c}$$

Thus, evaluation of the gradient of w on the boundary $y = 0$ requires the evaluation of a complex HSI of the type given in eqn (1). If we denote the limiting value of $w_{,x}(x,y) - iw_{,y}(x,y)$ as $z = x+iy \to x$ from Ω (i.e., $y \to 0^+$) to be $\sigma(x) - i\tau(x)$, then

$$\sigma(x) - i\tau(x) = \left[-\frac{1}{2\pi}P\int_{-\infty}^{\infty}\frac{\tau(\eta)}{\eta-x}\mathrm{d}\eta - i\frac{\tau(x)}{2}\right] + \lim_{y\to 0^+}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(\eta)}{(\eta-z)^2}\mathrm{d}\eta\right] \tag{18}$$

where the Plemelj formula was used to evaluate the first integral on the RHS of eqn (17).

Separating the real and imaginary parts of eqn (18) and using the *definition* eqn (9), $\tau(x)$ is:

$$\frac{\tau(x)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{w(\eta)}{(\eta-x)^2}\mathrm{d}\eta \tag{19}$$

If one considers the boundary value problem

$$w(|x| > 1, y = 0) = 0$$
$$w_{,y}(|x| < 1, y = 0) = p(x) \tag{20}$$

the integrand in eqn (19) vanishes outside the interval $[-1, 1]$ and one gets

$$\frac{\tau(x)}{2} = -\mathrm{Im}\left[\lim_{y\to 0^-} \frac{1}{2\pi i}\int_{-1}^{1} \frac{w(\eta)}{(\eta - z)^2}\,\mathrm{d}\eta\right] \equiv \frac{1}{2\pi}\int_{-1}^{1} \frac{w(\eta)}{(\eta - x)^2}\,\mathrm{d}\eta \tag{21}$$

Physically, the boundary condition eqn (20) corresponds to a finite Mode III crack in an infinite linear isotropic elastic solid with shear modulus of unity. The crack is located at $(|x| < 1, y = 0)$ with its tips at $x = \pm 1$. It is loaded by the traction $\tau(x) = -p(x)$ applied on the crack faces. From linear elastic fracture mechanics (LEFM), the shear stress component $\tau(x)$ directly ahead of the crack tip at $x = 1$ must have a square root singularity. This physical argument implies that the HSI

$$\frac{1}{2\pi i}\int_{-1}^{1} \frac{w(\eta)}{(\eta - x)^2}\,\mathrm{d}\eta$$

in general has singularities at its end points.

3. HYPERSINGULAR INTEGRALS (HSI) ON THE REAL LINE

Let $f(t)$ be a complex valued function (unless stated otherwise) of t. Let $f(t)$ be n times differentiable in (a, b) and $f^n(t)$ be Hölder continuous in (a, b). The complex valued HSI $I_n^{\pm}$ is defined as the limit of the analytic function $F_n(z)$

$$F_n(z) \equiv \frac{1}{2\pi i}\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}} \tag{22}$$

as $z = x + iy$ approaches the real axis $y = 0$ from the upper or lower half plane, i.e.,

$$I_n^{\pm}(x) \equiv \lim_{y\to 0^{\pm}} F_n(x+iy), \quad x \in (a, b) \tag{23}$$

The definition is consistent with the usage of HSI in the context of the boundary element method. Note that $F_n(z)$ is an analytic function in the complement of $[a, b]$ as long as $f(t)$ is continuous.

Proposition 1

$$I_n^{\pm}(x) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\pm\tfrac{1}{2}f(x) + \frac{1}{2\pi i}P\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-x)}\right] \tag{24}$$

where

$$P\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-x)}$$

is the principal value integral involving $f(t)$. A proof of this proposition is given in Appendix A.

Equation (24) shows that the value of the complex HSI depends on whether x is approached from the upper or lower half plane. Indeed, using eqn (24), one can easily show that

$$I_n^+(x) - I_n^-(x) = \frac{1}{n!}\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$$

$$I_n^+(x) + I_n^-(x) = \frac{1}{n!\pi i}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[P\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-x)}\right]. \tag{25}$$

Equation (25) a generalised version of the Plemelj formulas. However, if $f(t)$ is real valued, then the Imaginary part of $I_n^{\pm}(x)$ does not depend on whether x is approached from the upper or lower half plane. Indeed,

$$-\operatorname{Im}(I_n^{\pm}(x)) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{1}{2\pi}P\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-x)}\right] \tag{26}$$

Thus, one can define, for real $f(t)$ (see Introduction, eqns (6–9))

$$\frac{1}{2\pi}\int_a^b \frac{f(t)}{(t-x)^{n+1}}\,\mathrm{d}t \equiv -\operatorname{Im}\left[\lim_{y\to 0^\pm}\frac{1}{2\pi i}\int_a^b \frac{f(t)}{(t-z)^{n+1}}\,\mathrm{d}t\right] \tag{27}$$

Two examples, illustrating applications of eqn (24) are presented below. In these examples, we evaluate $I_n^{\pm}$ using the definition eqn (23). The result is then compared with eqn (24).

Example 1 : $f(t) = 2\pi i\, t^m$, m a positive integer, $a = 0, b = 1, n \geqslant 1$

Using eqn (24), $I_n^{\pm}(x)$ is found to be:

$$I_n^{\pm}(x) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\pm\pi i x^m + \sum_{k=1}^{m} C_k^m \frac{x^{m-k}}{k}[(1-x)^k - (-x)^k] + x^m \ln\left(\frac{1-x}{x}\right)\right] \tag{28}$$

where $C_k^m = m!/k!(m-k)!$. Note that $I_n(x) = I_n^{\pm}(x)$ if $n > m$. This function has non-integrable singularities at the end points $a = 0, b = 1$. The singularity at $b = 1$ is $O[(1-x)^{-n}]$ whereas the singularity at $a = 0$ is $O[x^{-(n-m)}]$ if $n > m$. For a special case, for example, $m = n = 1$, one gets, from eqn (27)

$$I_1^{\pm}(x) = \pm\pi i - \frac{x}{1-x} - 1 + \ln\frac{1-x}{x}, \quad x\in(0,1) \tag{29}$$

Direct evaluation of $F_1(z)$ (from eqn (22)) gives

$$F_1(z) = \ln\frac{z-1}{z} - \left(\frac{z}{1-z}+1\right), \quad z\notin(-\infty,\infty).$$

Taking the limit as z approaches the real axis from above and below (with the branch cut on [0, 1]) yields exactly the same result given by eqn (29).

Example 2 : consider the nontrivial case

$$f(t) = 2\pi i\frac{(1-t)^{\alpha-1}}{(1+t)^{\alpha}}, \quad 0 < \alpha < 1, \quad t\in(-1,1) \tag{30}$$

Let $a = -1, b = 1$ and $n = 1$ so that

$$F_1(z) = \int_{-1}^{1} \frac{(1-t)^{\alpha-1}\,dt}{(1+t)^{\alpha}(t-z)^2} \tag{31}$$

$I_1^{\pm}(x)$ is first calculated directly using the definition eqn (23). For $z \notin (-\infty,\infty)$, $F_1(z) = dG(z)/dz$ where

$$G(z) = \int_{-1}^{1} \frac{(1-t)^{\alpha-1}\,dt}{(1+t)^{\alpha}(t-z)} \tag{32}$$

It can be shown that (see Carrier *et al.*, 1966),

$$G(z) = -\frac{\pi(z-1)^{\alpha-1}}{\sin(\alpha\pi)(z+1)^{\alpha}} \tag{33}$$

so that

$$F_1(z) = -\frac{\pi(z-1)^{\alpha-1}}{\sin(\alpha\pi)(z+1)^{\alpha}}\left[\frac{\alpha-1}{z-1} - \frac{\alpha}{z+1}\right], \quad z \notin (-\infty,\infty) \tag{34}$$

Using eqn (34), with the branch cut on $[-1, 1]$,

$$I_1^{\pm} \equiv \lim_{y\to 0^{\pm}} F_1(x+iy) \quad \text{for} \quad x \in (-1, 1)$$

is found to be:

$$I_1^{\pm} \equiv \lim_{y\to 0^{\pm}} F_1(x+iy) = \frac{\pi(1-x)^{\alpha-1}\,e^{\pm i\alpha\pi}}{\sin(\alpha\pi)(x+1)^{\alpha}}\left[\frac{\alpha-1}{x-1} - \frac{\alpha}{x+1}\right], \quad x \in (-1, 1) \tag{35}$$

To check the validity of eqn (24), we first compute the principal value integral

$$P\int_{-1}^{1} \frac{(1-t)^{\alpha-1}\,dt}{(1+t)^{\alpha}(t-x)}$$

which is found to be (see Carrier *et al.*, 1966)

$$\pi\,\mathrm{cotan}(\alpha\pi)\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}} \tag{36}$$

Application of eqn (24) gives

$$I_1^{\pm}(x) = [\pm\pi i + \pi\,\mathrm{cotan}(\alpha\pi)]\frac{d}{dx}\left[\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}\right]. \tag{37}$$

The identities:

$$\pm\pi i + \pi\,\mathrm{cotan}(\alpha\pi) = \frac{\pi e^{\pm i\alpha\pi}}{\sin(\alpha\pi)} \tag{38}$$

and

$$\frac{d}{dx}\left[\frac{(1-x)^{\alpha-1}}{(1-x)^{\alpha}}\right]=\frac{(1-x)^{\alpha-1}}{(x+1)^{\alpha}}\left[\frac{\alpha-1}{x-1}-\frac{\alpha}{x+1}\right] \tag{39}$$

shows that eqn (35) and eqn (37) are the same.

Equation (37) can be generalised for the case of $n > 1$, i.e.,

$$I_n^{\pm}(x)=\left[\frac{\pm\pi i+\pi\,\mathrm{cotan}(\alpha\pi)}{n!}\right]\frac{d^n}{dx^n}\left[\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}\right] \tag{40}$$

Thus, the action of the operator defined by eqns (22–23) on

$$f(t)=2\pi i\left[\frac{(1-t)^{\alpha-1}}{(1+t)^{\alpha}}\right] \tag{41}$$

is precisely

$$(-1)^n\left[\frac{\pm\pi i+\pi\,\mathrm{cotan}(\alpha\pi)}{2\pi i n!}\right]\int_{-1}^{1}\delta_n(t-x)f(t)\,dt,$$

where δ_n is the nth derivative of the Dirac delta function!

The hypersingular integral $I_n^{\pm}(x)$ of

$$f(t)=2\pi i\frac{(1-t)^{\alpha-1}}{(1+t)^{\alpha}}P_m(t),\quad 0<\alpha<1,\quad t\in(-1,1)$$

where

$$P_m(t)\equiv\sum_{k=0}^{m}c_k t^k$$

is a mth order polynomial can be evaluated in exactly the same way. Indeed, following the procedure in Carrier *et al.* (1966), we found that

$$P\int_{-1}^{1}\frac{(1-t)^{\alpha-1}t^m\,dt}{(1+t)^{\alpha}(t-x)}=\pi\,\mathrm{cotan}(\alpha\pi)\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}x^m,\quad x\in(-1,1)$$

Application of (24) gives

$$I_n^{\pm}(x)=\frac{\pm\pi i+\pi\,\mathrm{cotan}(\alpha\pi)}{n!}\frac{d^n}{dx^n}\left[\frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha}}P_m(x)\right]$$

for all $x\in(-1,1)$. As before, the action of the operator defined by (23) on

$$f(t)=2\pi i\left[\frac{(1-t)^{\alpha-1}}{(1+t)^{\alpha}}P_m(t)\right]$$

is precisely

$$(-1)^n\left[\frac{\pm\pi i+\pi\cot\mathrm{an}(\alpha\pi)}{2\pi i n!}\right]\int_{-1}^{1}\delta_n(t-x)f(t)\,\mathrm{d}t.$$

Evaluation of the nth derivative

To evaluate

$$J_n\equiv\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[P\int_a^b\frac{f(t)\,\mathrm{d}t}{(t-x)}\right],$$

we shall assume that the $(n+1)$th derivative of f exists and is continuous in $[a,b]$, i.e., $f\in C^{n+1}[a,b]$. It is noted that

$$P\int_a^b\frac{f(t)\,\mathrm{d}t}{(t-x)}=-f(a)\ln(x-a)+f(b)\ln(b-x)-\int_0^{x-a}f'(x-\eta)\ln\eta\,\mathrm{d}\eta-\int_0^{b-x}f'(x+\eta)\ln\eta\,\mathrm{d}\eta \tag{42}$$

J_n can now be obtained by differentiating the above expression n times, i.e.,

$$J_n=\sum_{k=0}^{n-1}(-1)^{n-k}(n-k-1)!\left[\frac{f^k(a)}{(x-a)^{n-k}}-\frac{f^k(b)}{(x-b)^{n-k}}\right]+k_n \tag{43a}$$

where

$$k_n=f^n(b)\ln(b-x)-f^n(a)\ln(x-a)-\int_0^{x-a}f^{n+1}(x-\eta)\ln\eta\,\mathrm{d}\eta-\int_0^{b-x}f^{n+1}(x+\eta)\ln\eta\,\mathrm{d}\eta \tag{43b}$$

and $f^k(t)\equiv\mathrm{d}^kf(t)/\mathrm{d}t^k$. The condition $f\in C^{n+1}[a,b]$ can be relaxed if k_n in eqn (43) is replaced by

$$k_n=P\int_a^b\frac{f^n(t)\,\mathrm{d}t}{(t-x)}. \tag{43c}$$

In this case, $f\in C^{n,\alpha}[a,b]$, i.e., f is n times continuously differentiable in $[a,b]$ and f^n is Hölder continuous in $[a,b]$ with exponent $1>\alpha>0$.

4. RELATIONSHIP WITH HADAMARD FINITE PART INTEGRALS (HFP)

The HFP of

$$H_n(x)\equiv FP\int_a^b\frac{f(\eta)}{(\eta-x)^{n+1}}\mathrm{d}\eta,\quad x\in(a,b)$$

where f is a real valued function, was defined by Toh and Mukherjee (1994) as

$$H_n(x) \equiv \lim_{\varepsilon \to 0}\left[\int_{a-x}^{-\varepsilon} \frac{f(\eta-x)}{\eta^{n-1}}\,\mathrm{d}\eta + \int_{\varepsilon}^{b-x} \frac{f(\eta+x)}{\eta^{n+1}}\,\mathrm{d}\eta + \sum_{k=0}^{n-1} g(n,k)\frac{f^k(x)}{k!} + \int_{-\varepsilon}^{\varepsilon} \frac{1}{\eta^{n-1}}\left[f(\eta+x) - \sum_{k=0}^{n}\frac{f^k(x)}{k!}\eta^k\right]\mathrm{d}\eta\right] \quad (44a)$$

where

$$g(n,k) \equiv \frac{\varepsilon^{n+k}}{n-k}\left((-1)^{n-k}-1\right) \quad (44b)$$

and it is assumed that $f(t) \in C^n[a,b]$ and $f^n(x)$ is Hölder continuous in $[a,b]$. It should be noted that the last term in eqn (44a) is of $O(\varepsilon)$. In Appendix B, we show that this definition is consistent with our result, i.e.,

Proposition 2

The generalized HFP integral defined by eqn (44) is identical to

$$-\mathrm{Im}\left[\lim_{y\to 0}\frac{1}{i}\int_a^b \frac{f(\eta)}{(\eta-z)^{n+1}}\,d\eta\right],$$

which by eqn (24) is equivalent to:

$$H_n(x) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[P\int_a^b \frac{f(\eta)}{(\eta-x)}\,\mathrm{d}\eta\right] \equiv J_n(x) \quad (45)$$

provided that $f(t) \in C^{n+1}[a,b]$ and f is real valued. If eqn (43c) is used in the definition of $J_n(x)$, then the condition $f(t) \in C^{n+1}[a,b]$ can be replaced by $f \in C^{n,\alpha}[a,b]$.

It should be noted that the definition of HFP proposed by Toh and Mukherjee (1994) does not involve a limiting process, i.e., their definition of $H_n(x)$ is independent of ε_1 and ε_2 in their eqns (2–4). Equation (44) is obtained by taking $\varepsilon_1 = \varepsilon_2 = \varepsilon \to 0$ in their eqn (2). For the case of $n = 1$, $H_n(x)$ is equivalent to

$$H_1(x) = \lim_{\varepsilon\to 0}\left[\int_{a-x}^{-\varepsilon} \frac{f(\eta+x)}{\eta^2}\,\mathrm{d}\eta + \int_{\varepsilon}^{b-x} \frac{f(\eta+x)}{\eta^2}\,\mathrm{d}\eta - \frac{2f(x)}{\varepsilon}\right]$$

which is the definition used by Martin (1991) and Mangler (1952). Finally, eqn (45) implies that

$$H_{n+1} = \frac{1}{n+1}\frac{\mathrm{d}H_n(x)}{\mathrm{d}x} \quad (46)$$

Equation (46) is consistent with eqn (37) of Mangler (1952) who derived it using real variable calculus. Equations (45–46) are significant in view of eqns (20–22) of Toh and Mukherjee (1994) which state that the limit to the boundary of a real HSI is equal to its finite part.

Example

Consider the finite part integral

$$H_1(x) \equiv FP\int_0^1 \frac{t\,\mathrm{d}t}{(t-x)^2}, \quad x \in (0,1)$$

Using the formula given in eqn (45), with $f(t) = t$, $a = 0$, $b = 1$ and $n = 1$ gives the result

$$H_1(x) = -\frac{x}{1-x} - 1 + \ln\frac{1-x}{x}, \quad x \in (0,1)$$

This result is consistent with eqn (28) since one must take the real part of the right hand side of eqn (28).

5. LAPLACE EQUATION IN THE UHP REVISITED

We complete the earlier discussion on Laplace equation in the UHP. The starting point is eqn (18). Applying eqn (24) on the HSI in eqn (18), one gets

$$\lim_{y \to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(\eta)}{(\eta - z)^2}\mathrm{d}\eta = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{w(x)}{2} + \frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{w(\eta)}{\eta - x}\mathrm{d}\eta\right] \tag{47}$$

Now, from eqn (18),

$$\tau(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left[P\int_{-\infty}^{\infty}\frac{w(\eta)}{\eta - x}\mathrm{d}\eta\right] \tag{48a}$$

$$\sigma(x) = -\frac{1}{\pi}\left[P\int_{-\infty}^{\infty}\frac{\tau(\eta)}{\eta - x}\mathrm{d}\eta\right] \tag{48b}$$

Equations (48a, b) indicate that $\tau(x)$ and $\sigma(x)$ are not independent—a well known result in function theory.

Example

Consider the boundary value problem with $w(x, y = 0) \equiv w(x) = 1/(1+x^2)$. The exact solution of this problem can be obtained using Poisson's integral formulae, it is found to be:

$$w(x,y) = \frac{y+1}{(y+1)^2 + x^2}, \quad y > 0 \tag{49}$$

Equation (49) implies that

$$\frac{\partial w}{\partial y}(x, y = 0) \equiv \tau(x) = \frac{x^2 - 1}{(x^2+1)^2}. \tag{50}$$

Since

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta}{(x^2+1)(\eta - x)} = -\frac{x}{1+x^2} \tag{51}$$

by contour integration, eqn (48) requires that

$$\tau(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{1+x^2}\right] \quad \text{which is exactly} \quad \frac{x^2-1}{(x^2+1)^2}.$$

For the case of the loaded crack (with traction $\tau(x) = -p(x)$ on the crack faces) with tips at $x = \pm 1$, we have

$$-p(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left[P\int_{-1}^{1}\frac{w(\eta)}{\eta-x}\mathrm{d}\eta\right], \quad |x| < 1 \tag{52a}$$

For $|x| > 1$,

$$\tau(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-1}^{1}\frac{w(\eta)}{\eta-x}\mathrm{d}\eta\right], \quad |x| > 1 \tag{52b}$$

whereas

$$\sigma(x) = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\tau(\eta)\,\mathrm{d}\eta}{\eta-x} \tag{52c}$$

Example

Consider the special case $p(x) = p_o$, where p_o is a constant. The solution in this case is (Tada *et al.*, 1985):

$$w(x) = p_o\sqrt{1-x^2}, \quad |x| < 1 \tag{53}$$

Substituting (53) into eqn (52a) and evaluating the principal value integral gives

$$\frac{1}{\pi}P\int_{-1}^{1}\frac{w(\eta)\,\mathrm{d}\eta}{\eta-x} = -p_o x$$

so that

$$\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left[P\int_{-1}^{1}\frac{w(\eta)}{\eta-x}\mathrm{d}\eta\right] = -p_o$$

as required by the exact solution.

6. GENERALISATION OF HSI FOR CONTOURS

Equation (4) can be generalised to include simple smooth (open or closed) curves C in the (x, y) plane.

Proposition 3

Let $f(t) \in C^{n,\alpha}[C]$, then

$$I_n^{\pm} \equiv \lim_{z\to\zeta^{\pm}}\frac{1}{2\pi i}\int_C\frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}}$$

is given by

$$I_n^{\pm}(\zeta) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[\pm\tfrac{1}{2}f(\zeta) + \frac{1}{2\pi i}P\int_C \frac{f(t)\,\mathrm{d}t}{(t-\zeta)}\right] \tag{54}$$

where $\zeta \in C$. If C is open, then ζ cannot be an end point of C.

The proof is given in Appendix A. Upon completion of this work, the authors found out that for the case of closed contours, this theorem was proved earlier in the Russian literature by Yu. I. Cherskii (1959). His proof was summarised in the book titled *Boundary Value Problems* by Gakhov (1966) (English translation 1990). Indeed, Cherskii showed that the operations of differentiation and passing to the limit on the contour are interchangeable, i.e.,

$$\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[\frac{1}{2\pi i}P\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-\zeta)}\right] = \frac{1}{2\pi i}P\int_a^b \frac{f^n(t)\,\mathrm{d}t}{(t-\zeta)} \quad \text{(closed contour only)} \tag{55}$$

where $f^n(t)$ is the nth derivative of $f(t)$ along the contour C.

Example

Let $f(t) = 2\pi i$, $n = 1$, C is the part of the unit circle with center at the origin with end points at $z = 1$ and $z = i$.

In this case

$$F_1(z) = \int_C \frac{\mathrm{d}t}{(t-z)^2} = \frac{1}{1-z} - \frac{1}{i-z}$$

so that

$$I_1^{\mp}(\zeta) = \frac{1}{1-\zeta} - \frac{1}{i-\zeta}, \quad \zeta \in C, \quad \zeta \neq i \quad \text{or} \quad 1 \tag{56}$$

Note that in this case $I_1^+(\zeta) = I_1^-(\zeta)$.

$$\frac{1}{2\pi i}P\int_C \frac{f(t)\,\mathrm{d}t}{t-\zeta}$$

can be evaluated using contour integration, it is

$$\frac{1}{2\pi i}P\int_C \frac{f(t)\,\mathrm{d}t}{t-\zeta} = \ln(i-\zeta) - \ln(1-\zeta) + \pi i, \quad \zeta \in C, \quad \zeta \neq i \quad \text{or} \quad 1 \tag{57}$$

Equations (56) and (57) imply that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{1}{2\pi i}P\int_C \frac{f(t)\,\mathrm{d}t}{t-\zeta}\right] = I_1^{\mp}(\zeta),$$

which is eqn (54) for this particular $f(t)$.

7. CONCLUSIONS

The limit to the boundary approach, for the evaluation of HSI that occurs in 2-D BEM, is carried out in this work using complex variable techniques. These techniques allow us to derive an elegant and easy to use formula for the evaluation of such integrals. This formula can be applied to integrals on straight lines as well as on curved contours.

Toh and Mukherjee (1994), showed that the limit to the boundary of a HSI BEM integral is equal to its HFP. In this work, we have derived a new formula (eqn 45) that relates the HFP of a HSI to the derivative of a corresponding principal value integral. Furthermore, a very simple formula (eqn 46) is derived that relates HSI with different order of singularities.

The work presented here is expected to be of great value in BEM applications involving hypersingular integrals such as planar elasticity (for the evaluation of boundary stresses) and two dimensional fracture mechanics. Lee (1993), for example, demonstrates the role of integrals such as in our eqn (3) to represent stress functions in a complex BEM formulation for planar elasticity (see eqn (6) in Lee (1993)). Fracture mechanics problems present an additional challenge in that the square root stress singularity at a crack tip should be incorporated in an interpolation function. Integrals like in example 2 in Section 3 of this paper, containing products of polynomial functions and square root singularities at the end points (with $\alpha = 1/2$), should prove to be very valuable in such applications.

Overall, it is expected that practical numerical evaluation of hypersingular integrals in the BEM, using the formulae presented in this work, will be considerably simpler compared to the current state of the art where such integrals are generally evaluated using real variable calculus with suitable regularization and limiting processes.

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APPENDIX A

Proof of Proposition 1

Consider the function

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-z)} \tag{A1}$$

which is analytic everywhere except on the real axis. It is well known that (see Hille, 1973). for continuous $f(t)$, $\Phi(z)$ have derivatives of all orders in the region D outside $[a, b]$. In particular, $\Phi^n(z)$, the nth derivative of $\Phi(z)$ in D, is given by

$$\Phi^n(z) = \frac{n!}{2\pi i}\int_a^b \frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}} \tag{A2}$$

We first show that $\Phi^n(z)$ is continuous from the UHP (or from the LHP), i.e., $\Phi^n(z)$ tends to a definite limit $\Phi^n_-(x)$ (or $\Phi^n_-(x)$) which z approaches $x \in (a, b)$ along any path, which remains, however, on the UHP (or on the LHP). Consider any point $x \in (a, b)$ and let $N_x \subset (a, b)$ be a neighborhood about x. Then the integral in eqn (A2) can be written as the sum of two integrals, $\Phi^n_1(z)$ and $\Phi^n_2(z)$,

$$\Phi^n_1(z) = \frac{n!}{2\pi i}\int_{L_1} \frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}}, \quad \Phi^n_2(z) = \frac{n!}{2\pi i}\int_{L_2} \frac{f(t)\,\mathrm{d}t}{(t-z)^{n+1}} \tag{A3}$$

where $L_2 \equiv [a, b] - L_1$ and $N_x \subset L_1 = (c, d)$. Since $\Phi^n_2(z)$ is analytic in N_x, the problem is reduced to the study of $\Phi^n_1(z)$ as z approaches x from above or below.

Consider first the case $n = 1$. In this case

$$\Phi^1_1(z) = \frac{1}{2\pi i}\int_c^d \frac{f(t)\,\mathrm{d}t}{(t-z)^2} \tag{A4}$$

For $z \notin [c, d] = L_1$, we can integrate (A4) by parts since $f(t) \in C^{1,\alpha}$ resulting in

$$\Phi^1_1(z) = \frac{1}{2\pi i}\left[\frac{f(c)}{(c-z)} - \frac{f(d)}{(d-z)} + \int_c^d \frac{f'(t)\,\mathrm{d}t}{t-z}\right] \tag{A5}$$

Since $N_x \subset L_1$, the first two terms are analytic in N_x and therefore approach to a definite limit when z approaches $x \in N_x$. The problem is reduced to the study of

$$\psi(z) = \frac{1}{2\pi i}\int_c^d \frac{f'(t)\,dt}{t-z} \tag{A6}$$

as z approaches x from above or below. Note that $f'(t) \in C^{0,\alpha}$. This problem was studied by Muskhelishvili (1953), who proved that $\psi(z)$ is continuous from the UHP (or from the LHP), i.e., $\psi(z)$ tends to definite limit $\psi_+(x)$ (or $\psi_-(x)$) when z approaches $x \in (a, b)$ along any path, which remains, however, on the UHP (or on the LHP). Indeed, Muskhelishvili (1953) also showed that $\psi_+(x)$ and $\psi_-(x)$ satisfy the same Hölder condition as $f(x)$ if $\alpha < 1$. This proved that $\Phi^1(z)$ is continuous from the UHP (or from the LHP) and $I_1^{\pm} = \Phi^1_{\pm}(x) \equiv \lim_{y\to 0^{\pm}} \Phi^1(z)$ satisfied the Hölder condition as $f(x)$ for all $x \in (a, b)$. The proof of continuity for $n > 1$ proceeds in an identical way, i.e., we integrate by parts n times and use the condition that $f^n(t) \in C^{0,\alpha}$.

Now, the Cauchy Riemann equations imply that,

$$\Phi^n(z) = \frac{\partial^n \Phi}{\partial x^n} \tag{A7}$$

in D, which means that

$$I_n^{\pm}(x) \equiv \lim_{y\to 0^{\pm}} \frac{1}{2\pi i}\int_a^b \frac{f(t)\,dt}{(t-z)^{n+1}} = \frac{1}{n!}\lim_{y\to 0^{\pm}} \frac{\partial^n \Phi}{\partial x^n} = \frac{1}{n!}\frac{d^n}{dx^n}\left[\lim_{y\to 0^{\pm}} \Phi(x, y)\right] \tag{A8}$$

The proof is completed by noting that

$$\lim_{y\to 0^{\pm}} \Phi(x, y) = \pm\frac{f(x)}{2} + \frac{1}{2\pi i} P\int_a^b \frac{f(t)\,dt}{(t-x)}, \quad x \in (a, b) \tag{A9}$$

by the Plemelj formulas. In the last step of eqn (A8), we have interchanged the order of passing the limit and differentiation. This is intuitively obvious as y and x are independent and since both limit exist and are continuous on the boundary. A rigorous proof of the this result can be obtained as follows:

Consider first the case $n = 1$. We need to show that

$$\frac{d\Phi_+(x)}{dx} = \Phi^1_+(x) \equiv \lim_{y\to 0^+} \frac{1}{2\pi i}\int_a^b \frac{f(t)\,dt}{(t-z)^2}. \quad \text{Consider first } \frac{d\Phi_+}{dx},$$

we have, by definition

$$\frac{d\Phi_+}{dx} = \lim_{\delta\to 0} \frac{\Phi_+(x+\delta/2) - \Phi_+(x-\delta/2)}{\delta}$$

But

$$\Phi_+(x+\delta/2) - \Phi_+(x-\delta/2) = \int_\Gamma \Phi^1(t)\,dt = \Phi^1_+(x)\delta + \int_\Gamma [\Phi^1(t) - \Phi^1_+(x)]\,dt$$

where Γ is a semi-circular contour in the UHP with radius $\delta/2$ and center at x. Let ε be any real number > 0, we can select δ so that $|\Phi^1(t) - \Phi^1_+(x)| < \varepsilon$ for all t inside Γ, on account of the continuity of $\Phi^1(t)$ (from the UHP) as shown earlier. Thus

$$\left|\frac{\Phi_+(x+\delta/2) - \Phi_+(x-\delta/2)}{\delta} - \Phi^1_+(x)\right| < \varepsilon$$

and hence

$$\frac{d\Phi_{\pm}(x)}{dx} = \Phi^1_{\pm}(x) \equiv \lim_{y\to 0^{\pm}} \frac{1}{2\pi i}\int_a^b \frac{f(t)\,dt}{(t-z)^2}.$$

The proof for the case of $n > 1$ can be obtained by repeating the above reasoning.

Proof of Proposition 3

As before,

$$\Phi_n(z) = \frac{n!}{2\pi i}\int_C \frac{f(t)\,dt}{(t-z)^{n+1}}$$

is analytic outside C and is the nth derivative of

$$\Phi(z) \equiv \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-z)}.$$

For the case of closed contours, we can integrate by part n times, resulting in

$$\Phi_n(z) = \frac{n!}{2\pi i}\int_C \frac{f^n(t)\,dt}{(t-z)}$$

The previous arguments can be applied, almost word for word (with $\int_a^b$ replaced by $\int_C$ and the partial derivative $\partial/\partial x$ in (7) must be replaced by $d/d\zeta$ since the derivative of an analytic function is independent of direction). The result is:

$$I_n^{\pm}(\zeta) = \frac{1}{n!}\frac{d^n}{d\zeta^n}\left[\pm\tfrac{1}{2}f(\zeta) + \frac{1}{2\pi i}P\int_a^b \frac{f(t)\,dt}{(t-\zeta)}\right] \tag{A10}$$

APPENDIX B

Proof of Proposition 2

We want to show that

$$H_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left[P\int_a^b \frac{f(\eta)}{(\eta-x)}d\eta\right] \equiv J_n(x).$$

We prove this by mathematical induction. Using eqn (44), $H_1(x)$ is found to be

$$H_1(x) = \lim_{\varepsilon\to 0}\left[\frac{f(x-\varepsilon)+f(x+\varepsilon)-2f(x)}{\varepsilon} + \frac{f(a)}{a-x} - \frac{f(b)}{b-x} + L_\varepsilon\right] \tag{B1}$$

where

$$L_\varepsilon = \int_a^{x-\varepsilon} \frac{f'(\eta+x)\,d\eta}{\eta} + \int_\varepsilon^{b-x} \frac{f'(\eta+x)\,d\eta}{\eta} + O(\varepsilon) \tag{B2}$$

L_ε is, after one integration by parts,

$$L_\varepsilon = f'(b)\ln(b-x) - f'(a)\ln(x-a) - \int_a^{x-a} f''(x-\eta)\ln\eta\,d\eta - \int_\varepsilon^{b-x} f''(\eta+x)\ln\eta\,d\eta$$
$$+[f'(x-\varepsilon)-f'(x+\varepsilon)]\ln\varepsilon + O(\varepsilon) \tag{B3}$$

Taking $\varepsilon\to 0$ and comparing with eqns (43a, b) gives

$$H_1(x) = J_1(x) \tag{B4}$$

where we have used

$$\lim_{\varepsilon\to 0}\left[\frac{f(x-\varepsilon)+f(x-\varepsilon)-2f(x)}{\varepsilon}\right] = 0 \tag{B5}$$

$$\lim_{\varepsilon\to 0}[f'(x-\varepsilon)-f'(x+\varepsilon)]\ln\varepsilon = 0 \tag{B6}$$

The first two identities require $f\in C^{1,\alpha}[a,b]$ whereas the expression for L_ε requires $f\in C^2[a,b]$. Note that, if $f\in C^{1,\alpha}[a,b]$, we stop at eqn (b2) and $H_1(x) = J_1(x)$ if eqns (43a, c) are used.

Next, we show that if the proposition is true for $n > 1$, then it is true for $n+1$. To prove this we note that we need only to prove that

$$H_{n+1}(x) = \frac{1}{n+1}\frac{dH_n(x)}{dx} \tag{B7}$$

since by the induction hypothesis

$$H_n = J_n \quad \text{and} \quad J_k(x) \equiv \frac{1}{k!}\frac{d^k}{dx^k}\left[P\int_a^b \frac{f(\eta)}{(\eta-x)}d\eta\right].$$

Equation (B7) can be verified using the definition of H_n (eqn (44)) and the following properties of $g(n,k)$

For $n \neq k$,

(i) $g(n, k) = 0$ if n and k are both even or both odd

(ii) $g(n, k) = -2\varepsilon^{-n+k}/(n-k)$ otherwise

(iii) $g(n+1, k+1) = g(n, k)$

(iv) $g(n, k-1) = g(n+1, k)$

The usage of mathematical induction is not necessary. The validity of eqn (45) was also confirmed using repeated integration by parts as in the case of $n = 1$.